

MAXIMAL ABELIAN SUBGROUPS OF COMPACT MATRIX GROUPS

JUN YU

ABSTRACT. We classify closed abelian subgroups of the automorphism group of any compact classical simple Lie algebra whose centralizer has the same dimension as the dimension of the subgroup, and describe Weyl groups of maximal abelian subgroups.

Mathematics Subject Classification (2010). 20E07, 20E45, 20K27.

Keywords. Abelian subgroup, bimultiplicative function, Weyl group.

CONTENTS

1. Introduction	1
2. Projective unitary groups	3
3. Projective orthogonal groups	6
4. Projective symplectic groups	12
5. Twisted projective unitary groups	14
References	19

1. INTRODUCTION

In this paper, we study closed abelian subgroups F of a compact simple Lie group G satisfying the condition of

$$(*) \quad \dim \mathfrak{g}_0^F = \dim F,$$

where \mathfrak{g}_0 is the Lie algebra of G . It is clear that any maximal abelian subgroup of G satisfies the condition $(*)$. A good property of this class of abelian subgroups is the following: given a surjective homomorphism $p : G_1 \rightarrow G_2$ and a closed abelian subgroup F of G_1 , F satisfies the condition $(*)$ if and only if $p(F)$ satisfies the condition $(*)$. Precisely, we classify closed abelian subgroups of the automorphism group $G = \text{Aut}(\mathfrak{u}_0)$ satisfying the condition $(*)$ for \mathfrak{u}_0 a compact classical simple Lie algebra (except $\mathfrak{so}(8)$). In other publications, we classify closed abelian subgroups satisfying the condition $(*)$ of the automorphism group of any other compact simple Lie algebra.

The method of this paper is through linear algebra. We have four cases to consider: subgroups of the projective unitary group $\text{PU}(n)$; of the projective orthogonal group $\text{O}(n)/\langle -I \rangle$; of the projective symplectic group $\text{Sp}(n)/\langle -I \rangle$; and subgroups of $\text{PU}(n) \rtimes \langle \tau_0 \rangle$ (τ_0 = complex conjugation) not contained in $\text{PU}(n)$. The method of

classification for abelian subgroups of $\mathrm{PU}(n)$ is as follows. Given a closed abelian subgroup F of $\mathrm{PU}(n)$, we define an antisymmetric bimultiplicative function

$$m : F \times F \longrightarrow \mathrm{U}(1) = \{z \in \mathbb{C} : |z| = 1\}$$

by $m(x, y) = \lambda$ for any $x = [A], y = [B] \in F$, $A, B \in \mathrm{U}(n)$ with $ABA^{-1}B^{-1} = \lambda I$. Let $\ker m = \{x \in F : m(x, y) = 1, \forall y \in F\}$, which is a subgroup of F . By linear algebra, $(F/\ker m, m)$ determines and is determined by certain positive integers (n_1, n_2, \dots, n_s) with $n_{i+1} | n_i$ for any $1 \leq s-1$ and $n_1 n_2 \cdots n_s \mid n$. We show that these integers also determine the conjugacy class of F if it satisfies the condition (*). For a closed abelian subgroup F of $\mathrm{O}(n)/\langle -I \rangle$ (or $\mathrm{Sp}(n)/\langle -I \rangle$), similarly we define an antisymmetric bimultiplicative function $m : F \times F \longrightarrow \{\pm 1\}$. With this, we have a subgroup $\ker m$ of F . Moreover we define a subgroup B_F of $\ker m$, which is always a diagonalizable subgroup. Using it, F is expressed in a blockwise form. We are able to describe the image of the projection of F to each component if it satisfies the condition (*); and we describe the conjugacy class of F in terms of the integer $k = \frac{1}{2} \mathrm{rank}(F/\ker m)$, the conjugacy class of B_F and a combinatorial datum arising from m and the images of projections of F to block components. Such combinatorial data are generalization of symplectic metric spaces studied in [Yu], which arose from the study of elementary abelian 2-subgroups. Given a closed abelian subgroup F of $\mathrm{PU}(n) \rtimes \langle \tau_0 \rangle$ not contained in $\mathrm{PU}(n)$, we show that F lifts to a closed abelian subgroup F' of $(\mathrm{U}(n)/\langle -I \rangle) \rtimes \langle \tau \rangle$ with the conjugacy class of F and the conjugacy class of F' determine each other. Then, the classification is similar to the classification of closed abelian subgroups of $\mathrm{O}(n)/\langle -I \rangle$ satisfying the condition (*).

Given a compact simple Lie algebra \mathfrak{g}_0 with a complexification $\mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$, conjugacy classes of compact subgroups of $\mathrm{Aut}(\mathfrak{g})$ and that of $\mathrm{Aut}(\mathfrak{g}_0)$ are in one-to-one correspondence (cf. [AYY], Section 8). On the other hand, conjugacy classes of maximal compact abelian subgroups of $\mathrm{Aut}(\mathfrak{g})$ are in one-to-one correspondence with isomorphism classes of fine group gradings of \mathfrak{g} (cf. [EK], Section 2). The study of group gradings was initiated in [PZ89]. The fine group gradings of classical simple Lie algebras are classified in [BK] and [Eld10]. Our approach through the study of maximal abelian subgroups and the method employed here have several advantages. First the treatment is uniform, which enables us to show relations between abelian subgroups of different groups and similarity between the shape of the sets of abelian subgroups of them. Second the fusion of abelian subgroups are clearly understood from our classification, with which we are able to describe the Weyl groups $W(F) = N_G(F)/C_G(F)$ of maximal abelian subgroups F . In turn, our classification gives a better understanding of maximal abelian subgroups and the associated group gradings.

Notation and conventions. Given a Lie group G , write $Z(G)$ for the center of G and G_0 for the neutral component of G . Given a subgroup H of G , let $C_G(H)$ denote the centralizer of H in G and $N_G(H)$ denote the normalizer of H in G . For a subset X of G , let $\langle X \rangle$ denote the subgroup of G generated by elements in X . For a quotient group $G = H/N$, let $[x] = xN$ ($x \in H$) denote a coset. For a compact semisimple real Lie algebra \mathfrak{g}_0 , let $\mathrm{Aut}(\mathfrak{g}_0)$ be the group of automorphisms

of \mathfrak{g}_0 and $\text{Int}(\mathfrak{g}_0) = \text{Aut}(\mathfrak{g}_0)_0$ be the group of inner automorphisms. Denote by $Z_m = \{\lambda I_m : |\lambda| = 1\}$, which is the center of the unitary group $U(n)$. Let I_n be the $n \times n$ identity matrix. We define the following matrices,

$$I_{p,q} = \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix}, \quad J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \quad J'_n = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix},$$

$$K_n = \begin{pmatrix} 0 & 0 & 0 & I_n \\ 0 & 0 & -I_n & 0 \\ 0 & I_n & 0 & 0 \\ -I_n & 0 & 0 & 0 \end{pmatrix}.$$

2. PROJECTIVE UNITARY GROUPS

Let $\mathbb{R}, \mathbb{C}, \mathbb{H}$ be the set of real numbers, complex numbers and quaternion numbers respectively, which is either a field or a division ring. For $F = \mathbb{R}, \mathbb{C}$ or \mathbb{H} , let $M_n(F)$ be the set of $n \times n$ matrices with entries in F . Let

$$\begin{aligned} O(n) &= \{X \in M_n(\mathbb{R}) : XX^t = I\}, & SO(n) &= \{X \in O(n) : \det X = 1\}, \\ U(n) &= \{X \in M_n(\mathbb{C}) : XX^* = I\}, & SU(n) &= \{X \in U(n) : \det X = 1\}, \\ Sp(n) &= \{X \in M_n(\mathbb{H}) : XX^* = I\}. \end{aligned}$$

Here X^t denotes the transposition of a matrix X and X^* denotes the conjugate transposition of X . Defined as sets in this way, $O(n)$, $SO(n)$, $U(n)$, $SU(n)$, $Sp(n)$ are actually Lie groups, i.e., groups with a smooth manifold structure. Moreover, they are compact Lie groups, i.e., the underlying manifolds are compact. Also let $PO(n)$, $PSO(n)$, $PU(n)$, $PSU(n)$, $PSp(n)$ be the quotients of the groups $O(n)$, $SO(n)$, $U(n)$, $SU(n)$, $Sp(n)$ modulo their centers. Let

$$\begin{aligned} \mathfrak{so}(n) &= \{X \in M_n(\mathbb{R}) : X + X^t = 0\}, \\ \mathfrak{su}(n) &= \{X \in M_n(\mathbb{C}) : X + X^* = 0, \text{tr } X = 0\}, \\ \mathfrak{sp}(n) &= \{X \in M_n(\mathbb{H}) : X + X^* = 0\}, \end{aligned}$$

Then, $\mathfrak{so}(n)$, $\mathfrak{su}(n)$, $\mathfrak{sp}(n)$ are Lie algebras of $SO(n)$, $SU(n)$, $Sp(n)$ respectively. They represent all isomorphism classes of compact classical simple Lie algebras.

Let $G = PU(n) = U(n)/Z_n$, the projective unitary group of degree n . Let F be a closed abelian subgroup of G . For any $x, y \in F$, choose $A, B \in U(n)$ representing x, y . That is, $x = [A] = AZ_n$ and $y = [B] = BZ_n$. Since $1 = [x, y] = [A, B]Z_n$, $[A, B] = \lambda_{A,B}I$ for a complex number $\lambda_{A,B}$ with $|\lambda_{A,B}| = 1$. It is clear that the number $\lambda_{A,B}$ depends only on x, y , not on the choice of A and B . By this, we define a map $m : F \times F \longrightarrow U(1)$ by $m(x, y) = \lambda_{A,B}$. Since

$$1 = \det ABA^{-1}B^{-1} = \det(\lambda_{A,B}I) = (\lambda_{A,B})^n,$$

$m(x, y) = \lambda_{A,B} = e^{\frac{2k\pi i}{n}}$ for some integer k . The conclusion of the following lemma is clear.

Lemma 2.1. *The function m is antisymmetric and bimultiplicative. That means, $m(x, x) = 1$, $m(x, y) = m(y, x)^{-1}$ and $m(xy, z) = m(x, z)m(y, z)$ for any $x, y, z \in F$.*

Let $\ker m = \{x \in F : m(x, y) = 1, \forall y \in F\}$. It is a subgroup of F and the induced antisymmetric bimultiplicative function m on $F/\ker m$ is nondegenerate.

Lemma 2.2. *If F is a closed abelian subgroup of $\mathrm{PU}(n)$ satisfying the condition (*), then $\ker m = F_0$.*

Proof. Since m is a continuous map with finite image, one has $F_0 \subset \ker m$. For any $x \in \ker m$, substituting F by a subgroup conjugate to it if necessary, we may assume that $x = AZ_n$ for some $A = \mathrm{diag}\{\lambda_1 I_{n_1}, \lambda_2 I_{n_2}, \dots, \lambda_s I_{n_s}\}$, where $n_1, \dots, n_s \in \mathbb{Z}$, $n_1 + \dots + n_s = n$, and $\lambda_1, \dots, \lambda_s$ are distinct nonzero complex numbers. Since $x \in \ker m$, one has $F \subset \mathrm{U}(n)^A/Z_n$. From the condition of $\dim \mathfrak{g}_0^F = \dim F$, one gets $Z(\mathrm{U}(n)^A/Z_n)_0 \subset F_0$. Since $\mathrm{U}(n)^A = \mathrm{U}(n_1) \times \dots \times \mathrm{U}(n_s)$ and $Z(\mathrm{U}(n)^A/Z_n)_0 = (Z_{n_1} \times \dots \times Z_{n_s})/Z_n$, we get $x \in Z(\mathrm{U}(n)^A/Z_n)_0 \subset F_0$. \square

Given a positive integer k and a multiple $n = mk$ of k , define a subgroup H_k of $\mathrm{PU}(n)$ by

$$H_k = \langle [\mathrm{diag}\{I_m, \omega_k I_m, \dots, \omega_k^{k-1} I_m\}], \left[\begin{pmatrix} 0 & I_m & \cdots & 0 \\ 0 & 0 & \ddots & \ddots \\ \vdots & \ddots & \ddots & I_m \\ I_m & 0 & \cdots & 0 \end{pmatrix} \right] \rangle.$$

Proposition 2.1. *For a closed abelian subgroup F of G satisfying the condition (*), there exists positive integers $n_1 \geq n_2 \geq \dots \geq n_s \geq 2$ with $n_{i+1} | n_i$ for any $1 \leq i \leq s-1$ and $n_1 n_2 \cdots n_s | n$ such that F admits a decomposition $F = H_{n_1} \times \dots \times H_{n_s} \times T$, with T a torus of dimension $m-1$, where $m = \frac{n}{n_1 n_2 \cdots n_s}$. Moreover, the conjugacy class of F is uniquely determined by the positive integers (n_1, \dots, n_s) .*

Proof. We prove it by induction on the order of F/F_0 . If $|F/F_0| = 1$, i.e., F is connected, it must be a maximal torus of G , which corresponds to the case of $s = 0$ and $m = n$ in the conclusion. In general, choose $x_1, y_1 \in F$ such that $m(x_1, y_1)$ is of maximal order. Let $n_1 = o(m(x_1, y_1))$ and

$$F_1 = \{x \in F : m(x, x_1) = m(x, y_1) = 1\}.$$

We first show that: for any $x, y \in F$, $m(x, y)^{n_1} = 1$; there exists $x'_1 \in x_1 F_0$ and $y'_1 \in y_1 F_0$ such that $o(x'_1) = o(y'_1) = n_1$; for any of such x'_1, y'_1 , F admits a decomposition $F = \langle x'_1, y'_1 \rangle \times F_1$. Let $z = m(x_1, y_1)$. Since m is bimultiplicative and $m(x_1, y_1)$ is of maximal order, $\{m(x_1, \cdot)\} = \{m(\cdot, y_1)\} = \langle z \rangle$. Then, for any other $x \in F$, there exists integers a, b such that $m(x_1, x) = z^a$ and $m(x, y_1) = z^b$. Hence $x' = x x_1^{-b} y_1^{-a} \in F_1$. Therefore $F = \langle x_1, y_1 \rangle \cdot F_1$. Suppose there exists $x, y \in F$ such that $m(x, y)^{n_1} \neq 1$. Let $x = x_1^{a_1} y_1^{b_1} x_2$ and $y = x_1^{a_2} y_1^{b_2} y_2$ for some $a_1, a_2, b_1, b_2 \in \mathbb{Z}$ and $x_2, y_2 \in F_1$. Then $m(x, y) = m(x_1, y_1)^{a_1 b_2 - a_2 b_1} m(x_2, y_2)$. Hence $m(x_2, y_2)^{n_1} \neq 1$. Let $z' = m(x_2, y_2)$. Thus

$$m(x_1 x_2, y_1^a y_2^b) = m(x_1, y_1)^a m(x_2, y_2)^b = z^a z'^b.$$

Since $z'^{n_1} \neq 1$, we can choose $a, b \in \mathbb{Z}$ such that $o(z^a z'^b) > n_1$, which contradicts the assumption that $m(x_1, y_1)$ is of maximal order. Hence $m(x, y)^{n_1} = 1$ for any

$x, y \in F$. For any $y \in F$, we have $m(x_1^{n_1}, y) = m(x_1, y)^{n_1} = 1$ by the first statement proved above. Then, $x_1^{n_1} \in \ker m = F_0$ by Lemma 2.2. As F_0 is a torus, there exists $x'_1 \in x_1 F_0$ with $(x'_1)^{n_1} = 1$. Similarly there exists $y'_1 \in y_1 F_0$ with $(y'_1)^{n_1} = 1$. On the other hand, one has $n_1 | o(x'_1)$ and $n_1 | o(y'_1)$ since $m(x'_1, y'_1) = z$ has order n_1 . For any of such x'_1, y'_1 , $F = \langle x'_1, y'_1 \rangle \cdot F_1$ is proved above. If $x_1'^a y_1'^b \in F_1$ for some $a, b \in \mathbb{Z}$, then $1 = m(x_1'^a y_1'^b, y'_1) = z^a$. Hence $n_1 | a$. Similarly we have $n_2 | b$. Therefore $x_1'^a y_1'^b = 1$. By this we get $F = \langle x'_1, y'_1 \rangle \times F_1$. With this, we may assume that $o(x_1) = o(y_1) = o(x_1 F_1) = o(y_1 F_1) = n_1$ and $F = \langle x_1, y_1 \rangle \times F_1$. We may assume that $[x_1, y_1] = (\omega_{n_1})^{-1}$. Then, one can show that $(x_1, y_1) \sim ([A_1], [B_1])$, where

$$A_{n_1} = \text{diag}\{I_{m_1}, \omega_{n_1} I_{m_1}, \dots, \omega_{n_1}^{n_1-1} I_{m_1}\}$$

and

$$B_{n_1} = \begin{pmatrix} 0_{m_1} & I_{m_1} & 0_{m_1} & \cdots & 0_{m_1} \\ 0_{m_1} & 0_{m_1} & I_{m_1} & \cdots & 0_{m_1} \\ & & & \ddots & \\ 0_{m_1} & 0_{m_1} & 0_{m_1} & \cdots & I_{m_1} \\ I_{m_1} & 0_{m_1} & 0_{m_1} & \cdots & 0_{m_1} \end{pmatrix},$$

$m_1 = \frac{n}{n_1}$. Then, $F_1 \subset (\text{U}(n)^{\langle A_1, B_1 \rangle}) / Z_n \cong \text{U}(m_1) / Z_{m_1}$ and $|F_1 / (F_1)_0| = \frac{|F/F_0|}{n_1^2} < |F/F_0|$. By induction we finish the proof. One has $n_2 | n_1$ since $m(x, y)^{n_1} = 1$ for any $x, y \in F$. Similarly we have $n_{i+1} | n_i$ for any $1 \leq i \leq s-1$. \square

Given a sequence $\vec{a} = (n_1, \dots, n_s)$ with $n_{i+1} | n_i$ for any $1 \leq i \leq s-1$, let $V_{\vec{a}} = H_{n_1} \times \cdots \times H_{n_s}$ be an abelian subgroup with an antisymmetric bimultiplicative function induced from the functions on $\{H_{n_j} : 1 \leq j \leq s\}$ such that H_{n_i}, H_{n_j} are orthogonal for any $i \neq j$. Let $\text{Sp}(V_{\vec{a}})$ be the group of automorphisms of $V_{\vec{a}}$ preserving the function on it. Denote by

$$F_{\vec{a}, m} = H_{n_1} \times \cdots \times H_{n_s} \times T_m,$$

where $H_{n_i} \subset \text{PU}(n_i)$ as defined above and T_m is a maximal torus of $\text{PU}(m)$, $n = n_1 \times \cdots \times n_s \times m$, also we regard $\text{U}(n_1) \times \cdots \times \text{U}(n_s) \times \text{U}(m)$ as a subgroup of $\text{U}(n)$ by tensor product. By Proposition 2.1, any closed abelian subgroup of $\text{PU}(n)$ is conjugate to some $F_{\vec{a}, m}$. Apparently $F_{\vec{a}, m}$ is a maximal abelian subgroup of $\text{PU}(n)$.

Proposition 2.2. *One has $W(F_{\vec{a}, m}) = \text{Hom}(V_{\vec{a}}, \text{U}(1)^m / Z_m) \rtimes (S_m \times \text{Sp}(V_{\vec{a}}))$.*

Proof. Let $F = F_{\vec{a}, m}$. Since F_0 is stable under the action of $W(F)$ and the bimultiplicative function on F/F_0 is also preserved, we have a homomorphism $p : W(F) \longrightarrow \text{Aut}(F_0) \times \text{Sp}(V_{\vec{a}})$. Considering preservation of eigenvalues, one can show that $\text{Im } p \subset S_m \times \text{Sp}(V_{\vec{a}})$ and $\ker p = \text{Hom}(V_{\vec{a}}, \text{U}(1)^m / Z_m)$. Using $F = F' \times T_m$ with F' is a finite abelian subgroup of $\text{PU}(n/m)$ and T_m is a maximal torus of $\text{PU}(m)$, one sees that $\text{Im } p \supset S_m \times \text{Sp}(V_{\vec{a}})$. By this we reach the conclusion of the Proposition. \square

By Proposition 2.1, given two positive integers n, m with $m | n$, the group $\text{SU}(n) / \langle \omega_m I \rangle$ possesses a closed abelian subgroup satisfying the condition $(*)$ if and only if $n | m^k$ for some $k \geq 1$.

3. PROJECTIVE ORTHOGONAL GROUPS

Let $G = \mathrm{O}(n)/\langle -I \rangle$ and $\pi : \mathrm{O}(n) \rightarrow \mathrm{O}(n)/\langle -I \rangle$ be the natural projection. Given an abelian subgroup F of G , for any $x, y \in F$, choose $A, B \in \mathrm{O}(n)$ representing x, y . Since $1 = [x, y] = \pi([A, B])$, $[A, B] = \lambda_{A,B}I$ for some $\lambda_{A,B} = \pm 1$. It is clear that $\lambda_{A,B}$ depends only on x, y . We define a map $m : F \times F \rightarrow \{\pm 1\}$ by $m(x, y) = \lambda_{A,B}$. Then, m is an antisymmetric bimultiplicative function on F . Let $\ker m = \{x \in F : m(x, y) = 1, \forall y \in F\}$. As m is a continuous homomorphism taking values in ± 1 , one has $F_0 \subset \ker m$.

Lemma 3.1. *If F is a closed abelian subgroup of G satisfying the condition (*), then for any $x \in \ker m$, there exists $y \in F_0$ such that $xy \sim [I_{p,n-p}]$ for some integer p .*

Proof. Without loss of generality, we may assume that $x = [A]$ for some

$$A = \mathrm{diag}\{-I_p, I_q, A_{n_1}(\theta_1), \dots, A_{n_s}(\theta_s)\},$$

where $0 < \theta_1 < \dots < \theta_s < \pi$, $p + q + 2(n_1 + \dots + n_s) = n$,

$$A_k(\theta) = \begin{pmatrix} \cos(\theta)I_k & \sin(\theta)I_k \\ -\sin(\theta)I_k & \cos(\theta)I_k \end{pmatrix}.$$

Since $x \in \ker m$, we have

$$F \subset \mathrm{O}(n)^A / \langle -I \rangle = (\mathrm{O}(p) \times \mathrm{O}(q) \times \mathrm{U}(n_1) \times \dots \times \mathrm{U}(n_s)) / \langle -I \rangle.$$

From $\dim \mathfrak{g}_0^F = \dim F$, we get $\pi(1 \times 1 \times \mathrm{Z}_{n_1} \times \dots \times \mathrm{Z}_{n_s}) = Z(\mathrm{O}(n)^A / \langle -I \rangle)_0 \subset F_0$. Let $y = [\mathrm{diag}\{I_p, I_q, A_{n_1}(-\theta_1), \dots, A_{n_s}(-\theta_s)\}]$. Then $y \in F_0$ and $xy = [I_{p,n-p}]$. \square

Let

$$B_F = \{x \in \ker m : A^2 = 1, \forall A \in \pi^{-1}(x)\}.$$

For an even integer n , let

$$H_2 = \langle [I_{n/2, n/2}], [J'_{n/2}] \rangle \subset \mathrm{O}(n) / \langle -I \rangle.$$

Then,

$$(\mathrm{O}(n) / \langle -I \rangle)^{H_2} = \Delta(\mathrm{O}(n/2) / \langle -I \rangle) \times H_2,$$

where $\Delta(A) = \mathrm{diag}\{A, A\} \in \mathrm{O}(2m)$ for any $A \in \mathrm{O}(m)$. For an integer n with $4 \mid n$, let

$$H'_2 = \langle [J_{n/2}], [K_{n/4}] \rangle \subset \mathrm{O}(n) / \langle -I \rangle.$$

Then,

$$(\mathrm{O}(n) / \langle -I \rangle)^{H'_2} = \phi(\mathrm{Sp}(n/4) / \langle -I \rangle) \times H'_2,$$

where

$$\phi(A + \mathbf{i}B + \mathbf{j}C + \mathbf{k}D) = \begin{pmatrix} A & C & B & D \\ -C & A & D & -B \\ -B & -D & A & C \\ -D & B & -C & A \end{pmatrix}$$

for $A + \mathbf{i}B + \mathbf{j}C + \mathbf{k}D \in \mathrm{Sp}(n/4)$.

Lemma 3.2. *If F is a closed abelian subgroup of G satisfying the condition $(*)$ and with $B_F = 1$, then there exists $k \in \mathbb{Z}_{\geq 0}$ such that $F = (H_2)^k \times \Delta^k(F')$, where $n' = \frac{n}{2^k} = 1$ or 2 , $F' = \text{SO}(2)/\{\pm I\}$ if $n' = 2$, and $F' = 1$ if $n' = 1$.*

Proof. Since $B_F = 1$, by Lemma 3.1 one has $\ker m = F_0$. Then, the induced function m on F/F_0 is nondegenerate. Choose a finite subgroup F' of F such that $F = F' \times F_0$. Then the function m on F' is nondegenerate and we have

$$F_0 \subset (\text{O}(n)^{\pi^{-1}(F')})/\langle -I \rangle.$$

Since $x^2 \in \ker m = F_0$ for any $x \in F$, F' is an elementary abelian 2-group. Let $2k = \text{rank } F' = \text{rank } F/F_0$. By [Yu] Proposition 2.12, for a given integer k , the conjugacy class of F' has two possibilities. Precisely, let $\text{defe } F' - 1$ be the difference between the number of elements of F' conjugate to $\pi(I_{\frac{n}{2}, \frac{n}{2}})$ and the number of elements of F' conjugate to $\pi(J_{\frac{n}{2}})$. Then, $\text{defe } F' \neq 0$ and the conjugacy class of F' is determined by $\text{rank } F'$ and $\text{sign}(\text{defe } F')$.

If $\text{sign}(\text{defe } F') > 0$, then $F' = (H_2)^k$ and

$$\text{O}(n)^{\pi^{-1}(F')} = \Delta^k(\text{O}(n')) \times \pi^{-1}(F'),$$

where $n' = \frac{n}{2^k}$ and $\Delta^k = \Delta \circ \dots \circ \Delta$ (k -times). Since F satisfies the condition $(*)$, F_0 is a maximal torus of $\text{O}(n')/\langle -I \rangle$. Moreover, we have $n' = 1$ or 2 since otherwise F_0 has an element $[A]$ ($A \in \text{O}(n')$) with $A \neq \pm I$ and $A^2 = I$, which forces $B_F \neq 1$.

If $\text{sign}(\text{defe } F') < 0$, then $F' = (H_2)^{k-1} \times H'_2$ and

$$\text{O}(n)^{\pi^{-1}(F')} = \Delta^{k-1}(\text{Sp}(n')) \times \pi^{-1}(F'),$$

where $n' = \frac{n}{2^{k+1}}$, $\Delta^{k-1} = \Delta \circ \dots \circ \Delta$ ($(k-1)$ -times) and $\text{Sp}(n') \subset \text{O}(4n')$ is an inclusion given by the map ϕ described ahead of this lemma. Since F satisfies the condition $(*)$, F_0 is a maximal torus of $\text{Sp}(n')/\langle -I \rangle$. We must have $n' = 1$ since otherwise $B_F \neq 1$. On the other hand, when $n' = 1$, using an element in F_0 conjugate to $[i]$, we can find another finite subgroup F'' of F such that $F = F_0 \times F''$ and with $\text{defe } F'' > 0$. Therefore we return to the case of $\text{sign}(\text{defe } F') > 0$. \square

Proposition 3.1. *If F is a closed abelian subgroup of G satisfying the condition $(*)$, then there exists $k \geq 0$ and $s_0, s_1 \geq 0$ such that $n = 2^k s_0 + 2^{k+1} s_1$, $\dim F_0 = s_1$, $\text{rank}(F/\ker m) = 2k$ and $\text{rank}(\ker m/F_0) \leq \max\{s_0 - 1, 0\}$.*

Proof. Let $2k = \text{rank}(F/\ker m)$. Since $B_F \subset \ker m$, one has $F \subset \text{O}(n)^{\pi^{-1}(B_F)}/\langle -I \rangle$. By the definition of B_F , $\pi^{-1}(B_F)$ is an abelian subgroup of $\text{O}(n)$ with every element conjugate to $I_{p, n-p}$ for some p , $0 \leq p \leq n$. Hence $\pi^{-1}(B_F)$ is a diagonalizable subgroup of $\text{O}(n)$. Without loss of generality, we may assume that

$$\text{O}(n)^{\pi^{-1}(B_F)} = \text{O}(n_1) \times \dots \times \text{O}(n_s)$$

for some positive integers n_1, \dots, n_s with $n_1 + \dots + n_s = n$. Let $F_i \subset \text{O}(n_i)/\langle -I_{n_i} \rangle$ be the image of the projection of F to $\text{O}(n_i)/\langle -I_{n_i} \rangle$ and $p_i : F \rightarrow F_i$ be the projection. Each F_i as a subgroup of $\text{O}(n_i)/\langle -I_{n_i} \rangle$ has a bimultiplicative function m_i similar as the function m on F . An element $x \in F$ is of the form $x = [(A_1, \dots, A_s)]$, $A_i \in \text{O}(n_i)$ for any $1 \leq i \leq s$. For any other $y = [(B_1, \dots, B_s)] \in F$, by the

definition of the function m , one has $[A_i, B_i] = m(x, y)I_{n_i}$ for any i , $1 \leq i \leq s$. Hence $m_i(p_i(x), p_i(y)) = m(x, y)$ for any $x, y \in F$. We prove that $B_{F_i} = 1$ for any i . Suppose this fails. Then F has an element $x = [(A_1, \dots, A_s)]$ with $o(A_i) = 2$, $A_i \neq \pm I$ and $x_i = \pi(A_i) \in \ker m_i$. Since $x_i = \pi(A_i) \in \ker m_i$, by the equality $m_j(p_j(x), p_j(y)) = m(x, y)$ for any $y \in F$ and any $1 \leq j \leq s$, we get $x \in \ker m$. By the proof of Lemma 3.1, there exists $y = [(B_1, \dots, B_s)] \in F_0$ such that $B_i = I$ and $(A_j B_j)^2 = I$ for any $1 \leq j \leq s$. Hence $xy \in B_F$. On the other hand, since $B_i = I$ and $A_i \neq \pm I$,

$$O(n)^{\pi^{-1}(xy)} \not\supset O(n_1) \times \dots \times O(n_s).$$

This contradicts that $O(n)^{\pi^{-1}(B_F)} = O(n_1) \times \dots \times O(n_s)$.

For any $1 \leq i \leq s$, since $B_{F_i} = 1$, by Lemma 3.2, one has $n_i = 2^k n'_i$ with $n'_i = 1$ or 2 , and the conjugacy class of F_i is uniquely determined by the number k . Let s_0 be the number of indices i with $n'_i = 1$ and s_1 be the number of indices i with $n'_i = 2$. Then, $n = 2^k s_0 + 2^{k+1} s_1$. Without loss of generality we may assume that $n_i = 2^k$ if $1 \leq i \leq s_0$, and $n_i = 2^{k+1}$ if $s_0 + 1 \leq i \leq s$. Therefore

$$F_0 \subset O(n)^{\pi^{-1}(B_F)} / \langle -I \rangle = (O(n_1) \times \dots \times O(n_s)) / \langle (-I, \dots, -I) \rangle$$

is conjugate to $1^{s_0} \times \mathrm{SO}(2)^{s_1}$ if $s_0 > 0$, or to $\mathrm{SO}(2)^{s_1} / \langle (-I, \dots, -I) \rangle$ if $s_0 = 0$. Hence $\dim F = s_1$. Without loss of generality we may assume that $F_0 = 1^{s_0} \times \mathrm{SO}(2)^{s_1}$ if $s_0 > 0$, and $F_0 = \mathrm{SO}(2)^{s_1} / \langle (-I, \dots, -I) \rangle$ if $s_0 = 0$. Hence $B_F \cap F_0 = 1^{s_0} \times \{\pm 1\}^{s_1}$ if $s_0 > 0$, and $\{B_F \cap F_0 = \pm 1\}^{s_1} / \langle (-I, \dots, -I) \rangle$ if $s_0 = 0$. Therefore $\mathrm{rank}(B_F \cap F_0) = s_1 - \delta_{s_0, 0}$. Moreover

$$\begin{aligned} & \mathrm{rank}(\ker m / F_0) \\ &= \mathrm{rank}(B_F F_0 / F_0) \\ &= \mathrm{rank}(B_F / B_F \cap F_0) \\ &= \mathrm{rank} B_F - \mathrm{rank}(B_F \cap F_0) \\ &= \mathrm{rank} B_F - (s_1 - \delta_{s_0, 0}) \\ &\leq s_0 + s_1 - 1 - (s_1 - \delta_{s_0, 0}) \\ &= s_0 - 1 + \delta_{s_0, 0} \\ &= \max\{s_0 - 1, 0\}. \end{aligned}$$

□

As in the above proof, let F_i be the image of F under the projection $p_i : O(n)^{\pi^{-1}(B_F)} \rightarrow O(n_i) / \langle -I \rangle$. For $1 \leq i \leq s_0$, F_i is an elementary abelian 2-subgroup, which means $A_i^2 = \pm I$ if A_i is the i -th component of A for an element $x = [A] \in F$. Define $\mu_i : F \rightarrow \{\pm 1\}$ by $A_i^2 = \mu_i(x)I$. Since $p_i(\ker m) = 1$ if $1 \leq i \leq s_0$, μ_i descends to a map $F / \ker m \rightarrow \{\pm 1\}$. Moreover $p_i : F / \ker m \rightarrow F_i$ is an isomorphism transferring (m, μ_i) to (m_i, μ) . In this way, we get a linear structure $(m, \mu_1, \dots, \mu_{s_0})$ on $F / \ker m$. Note that, each μ_i is compatible with m , i.e., we have $m(x, y) = \mu_i(x)\mu_i(y)\mu_i(xy)$ for any $x, y \in F$; and as the proof of Lemma 3.2 shows, $(F / \ker m, m, \mu_i) \cong V_{0, k; 0, 0}$ for each $1 \leq i \leq s_0$ (cf. [Yu] Subsection 2.4).

Definition 3.1. A finite-dimensional vector space V over the field $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$ is called a multi-symplectic metric space if it is associated with a map $m : V \times V \longrightarrow \mathbb{F}_2$ and maps

$$\mu_1, \mu_2, \dots, \mu_s : V \longrightarrow \mathbb{F}_2$$

such that: $m(x, x) = 0$, $m(x, y) = m(y, x)$ and $m(x + y, z) = m(x, z) + m(y, z)$ for any $x, y, z \in V$; and $m(x, y) = \mu_i(x) + \mu_i(y) + \mu_i(xy)$ for any $x, y \in V$ and $1 \leq i \leq s$. Two multi-symplectic metric spaces $(V, m, \mu_1, \dots, \mu_s)$ and $(V', m', \mu'_1, \dots, \mu'_s)$ are called isomorphic if there exists a linear space isomorphism $f : V \longrightarrow V'$ transferring $(V, m, \mu_1, \dots, \mu_s)$ to $(V', m', \mu'_1, \dots, \mu'_s)$ for a permutation i_1, i_2, \dots, i_s of $1, 2, \dots, s$. Denote by $\text{Aut}(V, m, \mu_1, \dots, \mu_s)$ the group of linear isomorphisms $f : V \longrightarrow V$ which is also an isomorphism as multi-symplectic metric space.

Note that, in the above definition the order among $\mu_1, \mu_2, \dots, \mu_s$ is overlooked.

Proposition 3.2. The conjugacy class of a closed abelian subgroup F of G satisfying the condition $(*)$ is determined by the integer $k = \frac{1}{2} \text{rank}(F/\ker m)$, the conjugacy class of the subgroup B_F , and the linear structure $(m, \mu_1, \dots, \mu_{s_0})$ on $F/\ker m$.

Proof. Without loss of generality we may assume that

$$\text{O}(n)^{\pi^{-1}(B_F)} = \text{O}(n_1) \times \dots \times \text{O}(n_s)$$

for some positive integers n_1, \dots, n_s with $n_1 + \dots + n_s = n$; moreover, $n_i = 2^k$ if $1 \leq i \leq s_0$, and $n_i = 2^{k+1}$ if $s_0 + 1 \leq i \leq s$. By Lemma 3.2, $(F_i)_0 = 1$ if $1 \leq i \leq s_0$, and $(F_i)_0 \cong \text{SO}(2)/\langle -I \rangle$ if $s_0 + 1 \leq i \leq s$. We may assume that the subgroup B_F and the maps $\mu_i : F \longrightarrow \{\pm 1\}$, $1 \leq i \leq s_0$ are given. From these we determine the subgroup F up to conjugacy. The conjugacy class of each F_i is uniquely determined. The issue is how to match them. For an $x \in F$, let $x = [(A_1, \dots, A_{s_0}, A_{s_0+1}, \dots, A_s)]$, where $A_i \in \text{O}(n_i)$ for any $1 \leq i \leq s$. For $1 \leq i \leq s_0$, the conjugacy of A_i is determined by $\mu_i(x)$. For $s_0 + 1 \leq i \leq s$, as $F_i/(F_i)_0$ is an elementary abelian 2-group with a nondegenerate bimultiplicative function m_i and $(F_i)_0$ is a one-dimensional torus, we can modify A_i to make it conjugate to $I_{2^k, 2^k}$ or J_{2^k} . Inductively, we construct elements x_1, \dots, x_{2k} of F generating $F/\ker m$ with $m(x_{j_1}, x_{j_2}) = 1$ if and only if $\{j_1, j_2\} = \{2j-1, 2j\}$ for some $1 \leq j \leq k$, such that the conjugacy class of the tuple (x_1, \dots, x_{2k}) is determined by $\mu_1, \mu_2, \dots, \mu_{s_0}$. Since $F = \langle B_F, F_0, x_1, \dots, x_{2k} \rangle$, the conjugacy class of F is determined accordingly. \square

Proposition 3.3. If F is a closed abelian subgroup of G satisfying the condition $(*)$, then F is an elementary abelian 2-subgroup if and only if $s_0 = s$ and $\mu_i = \mu_j$ for any $1 \leq i, j \leq s_0$. If F is a maximal abelian subgroup, then $\text{rank}(\ker m/F_0) = \max\{s_0 - 1, 0\}$; if $\text{rank}(\ker m/F_0) = \max\{s_0 - 1, 0\}$ and $(s_0, s_1) \neq (2, 0)$, then F is a maximal abelian subgroup.

Proof. The first statement is clear. For the second statement, one has

$$F \subset (\text{O}(n)^{B_F}/\langle -I \rangle)^{F_0} \cong (\text{O}(2^k)^{s_0} \times \text{U}(2^k)^{s_1})/\langle (-I, \dots, -I) \rangle.$$

Hence $\{\pm I\}^{s_0} \times (\mathbb{Z}_{2^k})^{s_1}/\langle (-I, \dots, -I) \rangle \subset C_G(F)$. If F is a maximal abelian subgroup, then $\{\pm I\}^{s_0} \times (\mathbb{Z}_{2^k})^{s_1}/\langle (-I, \dots, -I) \rangle \subset F$. Furthermore one has

$$\{\pm I\}^{s_0} \times (\mathbb{Z}_{2^k})^{s_1}/\langle (-I, \dots, -I) \rangle \subset \ker m.$$

Thus $\text{rank}(\ker m/F_0) = \max\{s_0 - 1, 0\}$. Without loss of generality we may assume that $\pi^{-1}(F_0) = 1^{2^k s_0} \times \Delta^k(\text{SO}(2))^{s_1}$. Then,

$$C_G(F_0) = (\text{O}(2^k s_0) \times \text{U}(2^k)^{s_1}) / \langle (-I, \dots, -I) \rangle.$$

If $\text{rank}(\ker m/F_0) = \max\{s_0 - 1, 0\}$, we may and do assume that $B_F = \{\pm I_{2^k}\}^{s_0} \times (\pm I_{2^{k+1}})^{s_1}$. While $(s_0, s_1) \neq (2, 0)$ or $(0, 1)$, one has $n \neq 2^{k+1}$ and hence $I_{2^k, n-2^k} \not\sim -I_{2^k, n-2^k}$. Therefore

$$\begin{aligned} C_G(\ker m) &= (C_G(F_0))^{B_F} \\ &= ((\text{O}(2^k s_0) \times \text{U}(2^k)^{s_1}) / \langle (-I, \dots, -I) \rangle)^{B_F} \\ &= (\text{O}(2^k)^{s_0} \times \text{U}(2^k)^{s_1}) / \langle (-I, \dots, -I) \rangle. \end{aligned}$$

Since each F_i is a maximal abelian subgroup and the function m on $F/\ker m$ is nondegenerate, one has

$$\begin{aligned} C_G(F) &= ((\text{O}(2^k)^{s_0} \times \text{U}(2^k)^{s_1}) / \langle (-I, \dots, -I) \rangle)^F \\ &= \langle F, Z((\text{O}(2^k)^{s_0} \times \text{U}(2^k)^{s_1}) / \langle (-I, \dots, -I) \rangle) \rangle \\ &= \langle F, (\{\pm I_{2^k}\}^{s_0} \times (\Delta^k(\text{U}(1)))^{s_1}) / \langle (-I, \dots, -I) \rangle \rangle \\ &= \langle F, \ker m \rangle \\ &= F. \end{aligned}$$

Thus F is a maximal abelian subgroup. If $(s_0, s_1) = (0, 1)$, F is clearly a maximal abelian subgroup. \square

When $(s_0, s_1) = (2, 0)$, one can show that F is a maximal abelian subgroup if and only if it is not an elementary abelian 2-subgroup.

In the below we describe Weyl groups of maximal abelian subgroups of G . Given a maximal abelian subgroup F of G , by Proposition 3.1, we associate a function $m : F \times F \rightarrow \{\pm 1\}$, integers $k, s, s_0, s_1 = s - s_0$ with $n = 2^k s_0 + 2^{k+1} s_1$ and maps $\mu_1, \mu_2, \dots, \mu_{s_0} : F \rightarrow \{\pm 1\}$. Denote by $F_{k,m,\vec{\mu}}$ a maximal abelian subgroup of G like this, where $\vec{\mu}$ means the unordered tuple $(\mu_1, \mu_2, \dots, \mu_{s_0})$. For a map $\mu : F \rightarrow \{\pm 1\}$, let a_μ be the number of indices i , $1 \leq i \leq s_0$ such that $\mu_i = \mu$. Denote by $S_{\vec{\mu}} = \prod_\mu S_{a_\mu}$.

Proposition 3.4. *There is an exact sequence*

$$\begin{aligned} 1 &\rightarrow \text{Hom}(F/\ker m, B_F) \rtimes S_{\vec{\mu}} \rightarrow W(F_{k,m,\vec{\mu}}) \\ &\rightarrow \text{Aut}(F/\ker m, m, \vec{\mu}) \times (\{\pm 1\}^{s_1} \rtimes S_{s_1}) \\ &\rightarrow 1. \end{aligned}$$

Proof. Let $F = F_{k,m,\vec{\mu}}$. The induced action of $W(F)$ on $F/\ker m$ preserves m and $\vec{\mu}$. Hence there is a homomorphism $p : W(F) \rightarrow \text{Aut}(F/\ker m, m, \vec{\mu}) \times W(F_0)$, which is apparently a surjective map. It is clear that $W(F_0) \cong \{\pm 1\}^{s_1} \rtimes S_{s_1}$. There is another homomorphism $p' : \ker p \rightarrow W(\ker m)$. The image $p'(w)$ for an element $w \in \ker p$

is determined by the action of w on the first s_0 -components of B_F , which induces a permutation on the s_0 components isomorphic to $O(2^k)$ of $O(n)^{\pi^{-1}(B_F)}$ and hence a permutation of the indices $\{1, 2, \dots, s_0\}$, denoted by σ . Since w acts trivially on $F/\ker m$, $\mu_{\sigma(i)} = \mu_i$ for each $1 \leq i \leq s_0$. Therefore $\text{Im } p' \subset (\{\pm 1\}^{s_1} \rtimes S_{s_1}) \times \prod_{\mu} S_{a_{\mu}}$. It is clear that $\ker p' \subset \text{Hom}(F/\ker m, \ker m)$. Moreover considering the preservation of eigenvalues, one shows $\ker p' \subset \text{Hom}(F/\ker m, B_F)$. From the description of F as in the proof of Proposition 3.3, one can show that

$$\ker p = \text{Hom}(F/\ker m, B_F) \rtimes S_{\vec{\mu}}.$$

We reach the conclusion of the proposition. \square

For a general closed abelian subgroup F of G satisfying the condition $(*)$, $C_G(F)$ is not necessarily an abelian subgroup and the description of $W(F)$ is more complicated.

As an illustration of the classification of maximal abelian subgroups of $O(n)/\langle -I \rangle$, we classify multi-symplectic metric spaces while $s = 2$ or 3 .

Proposition 3.5. *Given a vector space V of dimension $2k$ over the field \mathbb{F}_2 with a nondegenerate bilinear form m , for any $k \geq 1$, there exist two isomorphism classes of multi-symplectic metric spaces (V, m, μ_1, μ_2) such that $(V, m, \mu_i) \cong V_{0,k;0,0}$, $i = 1, 2$; for any $k \geq 2$, there exist four isomorphism classes of multi-symplectic metric spaces $(V, m, \mu_1, \mu_2, \mu_3)$ such that $(V, m, \mu_i) \cong V_{0,k;0,0}$, $i = 1, 2, 3$.*

Proof. For $s = 2$, we have two cases to consider according to $\mu_1 = \mu_2$ or $\mu_1 \neq \mu_2$. While $\mu_1 = \mu_2$, there exists a unique isomorphism class since $(V, m, \mu_1) \cong V_{0,k;0,0}$. While $\mu_1 \neq \mu_2$, since μ_1, μ_2 are both compatible with m , $\mu_2 \mu_1^{-1} : V \rightarrow \mathbb{F}_2$ is a homomorphism. Let $V' = \ker(\mu_2 \mu_1^{-1})$. Then $V' \subset V$ is a subspace of codimension one. Thus $\text{rank}(\ker(m|_{V'})) = 1$ and $\text{defe}(V', \mu_1) = \frac{1}{2}(\text{defe}(V, \mu_1) + \text{defe}(V, \mu_2)) > 0$. Hence $(V', m|_{V'}, \mu_1|_{V'}) \cong V_{1,k-1;0,0}$ (cf. [Yu] Proposition 2.29). Therefore the isomorphism class of (V, m, μ_1, μ_2) is determined uniquely.

For $s = 3$, we have three cases to consider: $\mu_1 = \mu_2 = \mu_3$; $\mu_1 = \mu_2 \neq \mu_3$; $\mu_1 \neq \mu_2, \mu_3$ and $\mu_2 \neq \mu_3$. While $\mu_1 = \mu_2 = \mu_3$, there is a unique isomorphism class since $(V, m, \mu_1) \cong V_{0,k;0,0}$. While $\mu_1 = \mu_2 \neq \mu_3$, similarly as the above proof for $s = 2$ and in the case of $\mu_1 \neq \mu_2$, we get a unique isomorphism class. While $\mu_1 \neq \mu_2, \mu_3$ and $\mu_2 \neq \mu_3$, let $V' = \ker(\mu_2 \mu_1^{-1}) \cap \ker(\mu_3 \mu_1^{-1})$. Then V' is a subspace of codimension two of V . There are two cases according to $m|_{V'}$ is degenerate or not. In the case of $m|_{V'}$ is nondegenerate, let $V'' = \{x \in V : m(x, y) = 0, \forall y \in V'\}$. Then $V = V' \oplus V''$ and $m|_{V''}$ is also nondegenerate. Since $\mu_1|_{V'} = \mu_2|_{V'} = \mu_3|_{V'}$, one has $\text{defe}(V'', \mu_1|_{V''}) = \text{defe}(V'', \mu_2|_{V''}) = \text{defe}(V'', \mu_3|_{V''})$. If $\text{defe}(V'', \mu_1|_{V''}) < 0$, then $\mu_1|_{V''} = \mu_2|_{V''} = \mu_3|_{V''}$ and so $\mu_1 = \mu_2 = \mu_3$, which contradicts the assumption. If $\text{defe}(V'', \mu_1|_{V''}) > 0$, then $(\mu_1|_{V''}, \mu_2|_{V''}, \mu_3|_{V''})$ has only one possibility if the order among them is overlooked. Hence we get one isomorphism type of $(V, m, \mu_1, \mu_2, \mu_3)$ in this case. In the case of $m|_{V'}$ is degenerate, choose $V''' \subset V'$ such that $V' = V''' \oplus \ker(m|_{V'})$ and let $V'' = \{x \in V : m(x, y) = 0, \forall y \in V'''\}$. Then, $V = V''' \oplus V''$, $m|_{V'''}$, $m|_{V''}$ are nondegenerate, and $\dim V'' = 4$. One can show that $\text{defe } V''' > 0$ and $\text{defe}(\mu_1|_{V''}) > 0$. Moreover we can find generators x_1, y_1, x_2, y_2 of V'' such that

$V'' = \langle x_1, y_1 \rangle \oplus \langle x_2, y_2 \rangle$ as a symplectic vector space,

$$(\mu_1(x_1), \mu_1(x_2), \mu_1(y_1), \mu_1(y_2)) = (1, 1, 1, 1),$$

$$(\mu_2(x_1), \mu_2(x_2), \mu_2(y_1), \mu_2(y_2)) = (1, 1, 1, -1),$$

$$(\mu_3(x_1), \mu_3(x_2), \mu_3(y_1), \mu_3(y_2)) = (1, 1, -1, 1).$$

Thus we get one isomorphism class of $(V, m, \mu_1, \mu_2, \mu_3)$ in this case. \square

4. PROJECTIVE SYMPLECTIC GROUPS

Let $G = \mathrm{Sp}(n)/\langle -I \rangle$ and $\pi : \mathrm{Sp}(n) \rightarrow \mathrm{Sp}(n)/\langle -I \rangle$ be the natural projection. The classification of abelian subgroups of $\mathrm{Sp}(n)/\langle -I \rangle$ satisfying the condition $(*)$ is similar as the classification in the $\mathrm{O}(n)/\langle -I \rangle$ case. We give the main steps below, but omit some proofs. First we define a map $m : F \times F \rightarrow \{\pm 1\}$. For any $x = [A], y = [B] \in F$, if $[A, B] = \lambda_{A,B}I$, then $m(x, y) = \lambda_{A,B}$. Hence m is an antisymmetric bimultiplicative function on F . Let

$$\ker m = \{x \in F \mid m(x, y) = 1, \forall y \in F\}.$$

Lemma 4.1. *If F is a closed abelian subgroup of G satisfying the condition $(*)$, then for any $x \in \ker m$, there exists $y \in F_0$ such that $xy \sim [I_{p,n-p}]$ for some p , $0 \leq p \leq n$.*

Let

$$B_F = \{x \in \ker m : A^2 = 1, \forall A \in \pi^{-1}(x)\}.$$

It is a subgroup of $\ker m$. By Lemma 4.1, $\ker m = B_F F_0$. Let

$$H_2 = \langle [I_{n/2, n/2}], [J'_{n/2}] \rangle$$

and

$$H'_2 = \langle \mathbf{i}I, \mathbf{j}I \rangle.$$

Lemma 4.2. *If F is a closed abelian subgroup of G satisfying the condition $(*)$ and with $B_F = 1$, then either $n = 1$ and F is a maximal torus, or there exists $k \geq 1$ such that $F = (H_2)^{k-1} \times H'_2 \times \Delta^{k-1}(F')$, where $n' = \frac{n}{2^{k-1}} = 1$ or 2 , $F' = \mathrm{SO}(2)/\{\pm I\}$ if $n' = 2$, and $F' = 1$ if $n' = 1$.*

Proof. Let $2k = \mathrm{rank} F / \ker m$. By Lemma 4.1, $\ker m = B_F F_0 = F_0$ as we suppose that $B_F = 1$. If $k = 0$, then $F = F_0$ and it is a maximal torus of G . In this case $n = 1$ since it is assumed that $B_F = 1$. If $k \geq 1$, choosing a complement F' of F_0 in F , then F_0 is a maximal torus of $(\mathrm{Sp}(n)/\langle -I \rangle)^{F'}$. Since the bimultiplicative function m is nondegenerate on $F/\ker m = F/F_0$, it is nondegenerate on F' . By [Yu] Proposition 2.16, we have $F' = (H_2)^{k-1} \times H'_2$ or $F' = (H_2)^k$. Similarly as the proof of Lemma 3.2, in the second case, we can show that $\dim F > 0$. Choosing some $[A] \in F_0$ with $A \in \mathrm{Sp}(n)$ and $A^2 = -I$, we may replace F' by a different finite subgroup conjugate to $(H_2)^{k-1} \times H'_2$ and hence return to the first case. In the first case, we have $(\mathrm{Sp}(n)/\langle -I \rangle)^{F'} \cong \mathrm{O}(n/2^{k-1})/\langle -I \rangle$. Therefore $n/2^{k-1} = 1$ or 2 since F_0 is a maximal torus of $\mathrm{O}(n/2^{k-1})/\langle -I \rangle$ and it is assumed that $B_F = 1$. \square

Proposition 4.1. *If F is a closed abelian subgroup of G satisfying the condition $(*)$, then either F is a maximal torus of G , or there exists $k \geq 1$, $s_0, s_1 \geq 0$ such that $n = 2^{k-1}s_0 + 2^k s_1$, $\dim F_0 = s_1$, $\text{rank } F/\ker m = 2k$ and $\text{rank}(\ker m/F_0) \leq \max\{s_0 - 1, 0\}$.*

The proof is similar as that of Proposition 3.1. Note that we could regard the case of F being a maximal torus as the case of $k = 0$, $s_0 = 0$ and $s_1 = n$. By Proposition 4.1, any abelian subgroup of $\text{Sp}(n)$ satisfying the condition $(*)$ is a maximal torus; in particular $\text{Sp}(n)$ has no finite abelian subgroups satisfying the condition $(*)$.

Given an abelian subgroup F of G satisfying the condition $(*)$, the centralizer $\text{Sp}(n)^{\pi^{-1}(B_F)}$ possesses a blockwise decomposition

$$\text{Sp}(n)^{\pi^{-1}(B_F)} \sim \text{Sp}(n_1) \times \cdots \times \text{Sp}(n_s).$$

Let F_i be the image of F under the i -th projection $p_i : \text{Sp}(n)^{\pi^{-1}(B_F)} \rightarrow \text{Sp}(n_i)/\langle -I \rangle$. If F is not a maximal torus, by Lemma 4.2 and Proposition 4.1, we may assume that $n_i = 2^{k-1}$ if $1 \leq i \leq s_0$, and $n_i = 2^k$ if $s_0 + 1 \leq i \leq s$. For $1 \leq i \leq s_0$, define $\mu_i : F/\ker m \rightarrow \{\pm 1\}$ by $\mu_i(x) = \mu(p_i(x))$, $x \in F$. As $p_i(\ker m) = 1$, μ_i is well defined. Moreover $p_i : F/\ker m \rightarrow F_i$ is an isomorphism transferring (m, μ_i) to (m_i, μ) . In this way, we get a linear structure $(m, \mu_1, \dots, \mu_{s_0})$ on $F/\ker m$. Note that, each μ_i is compatible with m , i.e., $m(x, y) = \mu_i(x)\mu_i(y)\mu_i(xy)$ for any $x, y \in F$. Moreover by Lemma 4.2, one has $(F/\ker m, m, \mu_i) \cong V_{0, k-1; 0, 1}$ (cf. [Yu], Subsection 2.4) for each $1 \leq i \leq s_0$.

Proposition 4.2. *The conjugacy class of a closed abelian subgroup F of G satisfying the condition $(*)$ is determined by the integer $k = \frac{1}{2} \text{rank}(F/\ker m)$, the conjugacy class of the subgroup B_F , and the linear structure $(m, \mu_1, \dots, \mu_{s_0})$ on $F/\ker m$.*

The proof is along the same line as that of Proposition 3.2.

Proposition 4.3. *If F is a closed abelian subgroup of G satisfying the condition $(*)$, then it is an elementary abelian 2-subgroup if and only if $s_0 = s$ and $\mu_i = \mu_j$ for any $1 \leq i, j \leq s_0$. If F is a maximal abelian subgroup of G , then $\text{rank}(\ker m/F_0) = \max\{s_0 - 1, 0\}$; if $\text{rank}(\ker m/F_0) = \max\{s_0 - 1, 0\}$ and $(s_0, s_1) \neq (2, 0)$, then F is a maximal abelian subgroup.*

The proof is similar as that of Proposition 3.3.

The description of Weyl groups is also similar as orthogonal groups case. Given a maximal abelian subgroup F of G , by Proposition 4.1, we associate a function $m : F \times F \rightarrow \{\pm 1\}$, integers $k, s, s_0, s_1 = s - s_0$ with $n = 2^k s_0 + 2^{k+1} s_1$ and maps $\mu_1, \mu_2, \dots, \mu_{s_0} : F \rightarrow \{\pm 1\}$. Denote by $F_{k, m, \vec{\mu}}$ a maximal abelian subgroup of G like this, where $\vec{\mu}$ means the unordered tuple $(\mu_1, \mu_2, \dots, \mu_{s_0})$. For a map $\mu : F \rightarrow \{\pm 1\}$, let a_μ be the number of indices i , $1 \leq i \leq s_0$ such that $\mu_i = \mu$. Denote by $S_{\vec{\mu}} = \prod_\mu S_{a_\mu}$.

Proposition 4.4. *There is an exact sequence*

$$\begin{aligned} 1 &\longrightarrow \text{Hom}(F/\ker m, B_F) \rtimes S_{\vec{\mu}} \longrightarrow W(F_{k, m, \vec{\mu}}) \\ &\longrightarrow \text{Aut}(F/\ker m, m, \vec{\mu}) \times (\{\pm 1\}^{s_1} \rtimes S_{s_1}) \\ &\longrightarrow 1. \end{aligned}$$

5. TWISTED PROJECTIVE UNITARY GROUPS

For an integer $n \geq 2$, let $G = \mathrm{PU}(n) \rtimes \langle \tau_0 \rangle$, where $\tau_0 = \text{complex conjugation}$. Then, $\tau_0^2 = 1$ and $\tau_0([A])\tau_0^{-1} = [\overline{A}]$ for any $A \in \mathrm{U}(n)$. One knows that $\mathrm{Aut}(\mathfrak{su}(n)) \cong G$ if $n \geq 3$. Define $\overline{G} = \mathrm{U}(n) \rtimes \langle \tau \rangle$, $\tau^2 = 1$ and $\tau A \tau^{-1} = \overline{A}$ for any $A \in \mathrm{U}(n)$. Let $\pi : \overline{G} \rightarrow G$ be the adjoint homomorphism. Then $\ker \pi = Z_n$. Let $\tilde{G} = (\mathrm{U}(n)/\langle -I \rangle) \rtimes \langle \tau \rangle$ be a group with $\tau^2 = 1$ and $\tau[A]\tau^{-1} = [\overline{A}]$ for any $A \in \mathrm{U}(n)$. Let $\pi' : \tilde{G} \rightarrow G$ and $p : \overline{G} \rightarrow \tilde{G}$ be the natural projections. Then, $\pi' \circ p = \pi$.

Given an abelian subgroup F of G , let $H_F = F \cap G_0$. Define a map $m : H_F \times H_F \rightarrow \mathrm{U}(1)$ by $m(x, y) = \lambda$ if $x = \pi(A)$, $y = \pi(B)$, $A, B \in \mathrm{U}(n)$ and $[A, B] = \lambda I$. It is clear that m is well defined and it is an antisymmetric bimultiplicative function on H_F . Let

$$\ker m = \{x \in H_F : m(x, y) = 1, \forall y \in H_F\}.$$

Lemma 5.1. *If F is a closed abelian subgroup of G not contained in G_0 , then $m(x, y) = \pm 1$ for any $x, y \in H_F$, and $u^2 \in \ker m$ for any $u \in F - F \cap G_0$.*

Proof. For $x, y \in H_F$ and $u \in F - F \cap G_0$, let

$$x = \pi(A), \quad y = \pi(B), \quad u = \pi(C),$$

$A, B \in \mathrm{U}(n)$, $C \in \tau \mathrm{U}(n)$. Since F is abelian,

$$CAC^{-1} = \lambda_1 A, \quad CBC^{-1} = \lambda_2 B, \quad ABA^{-1}B^{-1} = \lambda I$$

for some complex numbers $\lambda_1, \lambda_2, \lambda \in \mathrm{U}(1)$. One has

$$\begin{aligned} \lambda I &= [A, B] = [\lambda_1 A, \lambda_2 B] \\ &= [CAC^{-1}, CBC^{-1}] = C[A, B]C^{-1} \\ &= C(\lambda I)C^{-1} = \lambda^{-1} I. \end{aligned}$$

Hence $m(x, y) = \lambda = \pm 1$. On the other hand,

$$\begin{aligned} C^2 B (C^2)^{-1} &= C(CBC^{-1})C^{-1} \\ &= C(\lambda_2 B)C^{-1} = C(\lambda_2 I)C^{-1}(CBC^{-1}) \\ &= (\lambda_2^{-1} I)(\lambda_2 B) = B, \end{aligned}$$

so $u^2 \in \ker m$. □

Choose an $u \in F - F \cap G_0$ and let $u = \pi(C)$, $C \in \tau \mathrm{U}(n)$. For any $A \in \pi^{-1}(\ker m)$, since F is abelian, $[C, A] = \lambda I$ for some complex number $\lambda \in \mathrm{U}(1)$. As we assume that $A \in \pi^{-1}(\ker m)$, λ does not depend on the choice of u and C . Let

$$\nu : \pi^{-1}(\ker m) \rightarrow \mathrm{U}(1)$$

be defined by $\nu(A) = \lambda$. It is a group homomorphism.

Lemma 5.2. *The map $\pi : \ker \nu / \langle -I \rangle \rightarrow \ker m$ is an isomorphism.*

Proof. For any $x \in \ker m$, choosing $A \in \pi^{-1}(x)$, then $[C, A] = \lambda^2 I$ for some $\lambda \in \mathrm{U}(1)$. By this one has $[C, \lambda A] = [C, \lambda I][C, A] = \lambda^{-2} \lambda^2 I = I$. Thus $\lambda A \in \pi^{-1}(x) \cap \ker \nu$ and hence $\pi : \ker \nu / \langle -I \rangle \rightarrow \ker m$ is surjective. On the other hand, if $\pi([A]) =$

1, then $A = \lambda I$ for some $\lambda \in \mathbf{U}(1)$ and $[C, A] = I$. Thus $I = [C, A] = [C, \lambda I] = \lambda^{-2}I$. Therefore $A = \pm I$ and hence $\pi : \ker \nu / \langle -I \rangle \rightarrow \ker m$ is injective. \square

Let $B_F = \{A \in \ker \nu : A^2 = I\}$. Note that, here we define B_F as a subgroup of \overline{G} , rather than a subgroup of G . The following lemma indicates that $\ker m = \pi(B_F)F_0$.

Lemma 5.3. *If F is a closed abelian subgroup of G satisfying the condition $(*)$ and being not contained in G_0 , then for any $x \in \ker m$, there exists $y \in F_0$ such that $xy \in \pi(B_F)$.*

Proof. Choose an $A \in \ker \nu \cap \pi^{-1}(x)$. Since $A \in \ker \nu$, one has $[C, A] = I$. Hence A and $\tau A \tau^{-1} = \overline{A}$ are similar matrices. By this the multiplicity of an eigenvalue λ of A is equal to the multiplicity of the eigenvalue $\overline{\lambda}$ of A . We may assume that

$$A = \text{diag}\{-I_p, I_q, A_{n_1}(\theta_1), \dots, A_{n_s}(\theta_s)\},$$

where $0 < \theta_1 < \dots < \theta_s < \pi$, $p + q + 2(n_1 + \dots + n_s) = n$,

$$A_k(\theta) = \begin{pmatrix} \cos(\theta)I_k & \sin(\theta)I_k \\ -\sin(\theta)I_k & \cos(\theta)I_k \end{pmatrix}.$$

Since $A \in \ker \nu$, one has

$$\pi^{-1}(F) \subset \overline{G}^A = (\mathbf{U}(p) \times \mathbf{U}(q) \times L_1 \times \dots \times L_s) \rtimes \tau,$$

where

$$L_i = \left\{ \begin{pmatrix} X & Y \\ -Y & X \end{pmatrix} \in \mathbf{U}(2n_i) : X, Y \in \mathbf{M}(n_i, \mathbb{C}) \right\},$$

$1 \leq i \leq s$. It holds that $(\overline{G}^A)_0 \supset 1 \times 1 \times Z'_{n_1} \times \dots \times Z'_{n_s}$, where

$$Z'_{n_i} = \left\{ \begin{pmatrix} \cos(\theta)I_{n_i} & \sin(\theta)I_{n_i} \\ -\sin(\theta)I_{n_i} & \cos(\theta)I_{n_i} \end{pmatrix} : 0 \leq \theta \leq 2\pi \right\}.$$

Since F satisfies the condition $(*)$, one has $\pi((\overline{G}^A)_0) \subset F_0$. Let

$$y = \pi(\text{diag}\{I_p, I_q, A_{n_1}(-\theta_1), \dots, A_{n_s}(-\theta_s)\}).$$

Then $y \in F_0$ and $xy \in \pi(B_F)$. \square

The following lemma is well known, a proof of it could be found in [BtD], Page 177.

Lemma 5.4. *Let G be a compact (not necessarily connected) Lie group and $x \in G$ be an element. If T is a maximal torus of $(G^x)_0$, then any other element y of xG_0 is conjugate to an element in xT and any maximal torus of $(G^y)_0$ is conjugate to T .*

For an even integer n , let $H_2 = \langle [I_{n/2, n/2}], [J'_{n/2}] \rangle \subset \mathbf{PU}(n)$. Then,

$$((\mathbf{U}(n)/\mathbf{Z}_n) \rtimes \langle \tau_0 \rangle)^{H_2} = (\Delta(\mathbf{U}(n/2)/\mathbf{Z}_{n/2}) \rtimes \langle \tau_0 \rangle) \times H_2,$$

where $\Delta(A) = \text{diag}\{A, A\} \in \mathbf{U}(2m)$ for any $A \in \mathbf{U}(m)$.

Lemma 5.5. *Let F be a closed abelian subgroup of G satisfying the condition $(*)$ and being not contained in G_0 . If $B_F = \{\pm I\}$, then there exists $k \in \mathbb{Z}_{\geq 0}$ such that $F = (H_2)^k \times \Delta^k(F')$, where F' is a maximal torus of $(U(n')/Z_{n'}) \rtimes \langle \tau_0 \rangle$ and $n' = \frac{n}{2^k} = 1$ or 2 .*

Proof. Since $B_F = \{\pm I\}$, one has $\ker m = F_0$ by Lemma 5.3. Then, the induced function m on H_F/F_0 is nondegenerate. Let $2k = \text{rank}(H_F/\ker m)$. Choosing a finite subgroup F' of H_F such that $H_F = F' \times F_0$, then the function m on F' is nondegenerate. Thus F' is an elementary abelian 2-group by Lemma 5.1. By [Yu], Proposition 2.4, any non-identity element of F' is conjugate to $\pi(I_{\frac{n}{2}, \frac{n}{2}})$ and the conjugacy class of F' is uniquely determined by $k = \frac{1}{2} \text{rank } F'$. Hence $F' \sim (H_2)^k$. Substituting F by a subgroup conjugate to it if necessary, we may assume that

$$C_G(F') = F' \times \Delta^k((U(n')/Z_{n'}) \rtimes \langle \tau_0 \rangle),$$

where $n' = \frac{n}{2^k}$. Then, there exists $\tau' \in F - H_F$ such that

$$\langle F_0, \tau' \rangle \subset (U(n')/\langle Z_{n'} \rangle) \rtimes \langle \tau_0 \rangle.$$

Since F satisfies the condition $(*)$, F_0 is a maximal torus of $(U(n')/Z_{n'})^{\tau'}$. By Proposition 5.4, F_0 is conjugate to a maximal torus of $\text{SO}(n')$ (if $2 \nmid n$) or $\text{SO}(n')/\langle -I \rangle$ (if $2 \mid n'$). As it is assumed that $B_F = \{\pm I\}$, one has $n' = 1$ or 2 . We reach the conclusion of the proposition. \square

Lemma 5.6. *If F is an abelian subgroup of G not contained in G_0 , then there exists an abelian subgroup F' of \tilde{G} such that $\pi'(F') = F$ and $F' \cap (Z_n/\langle -I \rangle) = \langle iI \rangle/\langle -I \rangle$. Given F , F' is determined up to conjugation by an element in $Z_n/\langle -I \rangle$.*

Proof. By Lemma 5.3, $m(x, y) = \pm 1$ for any $x, y \in H_F$. Hence $\pi'^{-1}(H_F)$ is an abelian subgroup of $U(n)/\langle -I \rangle$. Choose any $u \in \pi'^{-1}(F - H_F)$ and let

$$F' = \langle (\pi'^{-1}(H_F))^u, u \rangle \subset (U(n)/\langle -I \rangle) \rtimes \langle \tau \rangle.$$

For any $x' \in \pi'^{-1}(H_F)$, since F is abelian, one has $ux'u^{-1}x'^{-1} = [\lambda^2 I]$ for some $\lambda \in U(1)$. Then $u(\lambda x')u^{-1} = \lambda x'$ and hence $\lambda x' \in F'$. Therefore $\pi(F') = F$. If $[\lambda I] \in F' \cap (Z_n/\langle -I \rangle)$, then $[\lambda I] = u[\lambda I]u^{-1} = [\lambda^{-1} I]$. Thus $[\lambda I] = 1$ or $[iI]$ and hence $F' \cap (Z_n/\langle -I \rangle) = \langle iI \rangle/\langle -I \rangle$. For a subgroup F' of $(U(n)/\langle -I \rangle) \rtimes \langle \tau \rangle$ satisfying the conditions in the proposition, choose $x \in F - H_F$ and $u \in F' \cap \pi'^{-1}(x)$. Since $[\lambda I]u[\lambda I]^{-1} = [\lambda^2 I]u$, the conjugacy class of u is determined up to conjugation by an element in $Z_n/\langle -I \rangle$. Fixing u , for any $y \in H_F$ and $y' \in \pi'^{-1}(y)$, if y' and $[\lambda I]y'$ both commute with u , then

$$\begin{aligned} & 1 \\ &= u([\lambda I]y')u^{-1}([\lambda I]y')^{-1} \\ &= (u[\lambda I]u^{-1}[\lambda I]^{-1})[\lambda I](uy'u^{-1}y'^{-1})[\lambda I]^{-1} \\ &= [\lambda^{-2} I][\lambda I][\lambda I]^{-1} \\ &= [\lambda^{-2} I]. \end{aligned}$$

Hence $[\lambda I] = 1$ or $[iI]$. As $[iI] \in F'$, F' is determined by u . Therefore the conjugacy class of F' is determined up to conjugation by an element in $Z_n/\langle -I \rangle$. \square

In this correspondence, one can show that F satisfies the condition $(*)$ if and only if F' satisfies it; F is a maximal abelian subgroup if and only if F' is. Denote by $H_{F'} = F' \cap \tilde{G}$.

Given a closed abelian subgroup F of G satisfying the condition $(*)$ and being not contained in G_0 , by Lemma 5.6, F lifts to an abelian subgroup F' of $\tilde{G} = (\mathrm{U}(n)/\langle -I \rangle) \rtimes \langle \tau \rangle$ satisfying the condition $(*)$ and being not contained in \tilde{G}_0 . We define an antisymmetric bimultiplicative function $m' : H_{F'} \times H_{F'} \rightarrow \{\pm 1\}$ and a homomorphism $\nu' : \ker m' \rightarrow \{\pm 1\}$ by $[A, B] = m'(x, y)I$, $[C, A'] = \nu'(x')I$ for $x = [A], y = [B] \in H_{F'}$, $x' = [A'] \in \ker m'$, $u = [C] \in F' - H_{F'}$. Note that $\nu'([iI]) = -1$. Hence

$$\ker m' = \ker \nu' \times \langle [iI] \rangle.$$

Let $B_{F'} = \{x \in \ker \nu' : A^2 = I, \forall A \in p^{-1}(x)\}$. Similarly as Lemma 5.3, one can show that

$$\ker \nu' = B_{F'} F_0.$$

It is clear that $B_{F'} = p(B_F)$. The following lemma is analogous to Lemma 5.5, which follows from Lemmas 5.5 and 5.6.

Lemma 5.7. *Let F' be a closed abelian subgroup of \tilde{G} satisfying the condition $(*)$ and being not contained in \tilde{G}_0 . If $B_{F'} = 1$, then there exists $k \in \mathbb{Z}_{\geq 0}$ such that $F = (H_2)^k \times \langle [iI] \rangle \times \Delta^k(F')$, where $n' = \frac{n}{2^k} = 1$ or 2 , $F' = \mathrm{SO}(2)/\{\pm I\} \times \langle \tau_0 \rangle$ if $n' = 2$, and $F' = \langle \tau_0 \rangle$ if $n' = 1$.*

Without loss of generality, we may assume that

$$(\overline{G})^{B_F} = (\mathrm{U}(n_1) \times \cdots \times \mathrm{U}(n_s)) \rtimes \langle \tau \rangle$$

for some positive integers n_1, \dots, n_s with $n_1 + \cdots + n_s = n$. Let F'_i be the image of the projection of F' to $(\mathrm{U}(n_i)/\langle -I \rangle) \rtimes \langle \tau \rangle$ and $p_i : F' \rightarrow (\mathrm{U}(n_i)/\langle -I \rangle) \rtimes \langle \tau \rangle$ be the projection map. One can show that $B_{F'_i} = 1$ for each i , $1 \leq i \leq s$. Let $k = \frac{1}{2} \mathrm{rank}(H_{F'}/\ker m')$. By Lemma 5.7, $n_i = 2^k$ or 2^{k+1} . We may assume that $n_1 = n_2 = \cdots = n_{s_0} = 2^k$, $n_{s_0+1} = \cdots = n_s = 2^{k+1}$. For $1 \leq i \leq s_0$, define $\mu_i : H_{F'}/\ker \nu' \rightarrow \{\pm 1\}$ by $\mu_i(x) = \mu(p_i(x))$. That is, for $x = [(A_1, A_2, \dots, A_s)] \in H_{F'}$, one has $A_i^2 = \mu_i(x)I$ for $1 \leq i \leq s_0$. As $p_i(\ker \nu') = 1$, μ_i is well defined. Moreover $p_i : H_{F'}/\ker \nu' \rightarrow H_{F'_i}$ is an isomorphism transferring (m, μ_i) to (m_i, μ) . In this way, we define a linear structure $(m, \mu_1, \dots, \mu_{s_0})$ on $H_{F'}/\ker \nu'$. Note that, each μ_i is compatible with m' , i.e.,

$$m'(x, y) = \mu_i(x)\mu_i(y)\mu_i(xy)$$

for any $x, y \in H_{F'}$. By Lemma 5.7, as a symplectic metric space one has (cf. [Yu], Subsection 2.4)

$$(H_{F'}/\ker \nu', m, \mu_i) \cong V_{0,k;1,0}$$

for any i , $1 \leq i \leq s_0$.

Proposition 5.1. *The conjugacy class of a closed abelian subgroup F' of \tilde{G} satisfying the condition $(*)$ is determined by the integer $k = \frac{1}{2} \mathrm{rank}(H_{F'}/\ker m')$, the conjugacy class of the subgroup $B_{F'}$, and the linear structure $(m', \mu_1, \dots, \mu_{s_0})$ on $H_{F'}/\ker \nu'$.*

Proof. Without loss of generality we may assume that

$$\overline{G}^{B_F} = (\mathrm{U}(n_1) \times \cdots \times \mathrm{U}(n_s)) \rtimes \langle \tau \rangle$$

for some positive integers n_1, \dots, n_s with $n_1 + \cdots + n_s = n$; moreover, $n_i = 2^k$ if $1 \leq i \leq s_0$, and $n_i = 2^{k+1}$ if $s_0 + 1 \leq i \leq s$, where $F = \pi'(F')$. By Lemma 5.5, $(F'_i)_0 = 1$ if $1 \leq i \leq s_0$, and $(F'_i)_0 \cong \mathrm{SO}(2)/\langle -I \rangle$ if $s_0 + 1 \leq i \leq s$. We may assume that the subgroup $B_{F'}$ and the maps $\mu_i : H_{F'}/\ker \nu' \rightarrow \{\pm 1\}$, $1 \leq i \leq s_0$ are given. From these we determine the subgroup F' up to conjugacy. The conjugacy class of each F'_i is uniquely determined by Lemma 5.7. The issue is how to match them. For an $x \in H_{F'}$, let $x = [(A_1, \dots, A_{s_0}, A_{s_0+1}, \dots, A_s)]$, where $A_i \in \mathrm{U}(n_i)$, $1 \leq i \leq s$, we have $A_i^2 = \mu_i(x)I$ if $1 \leq i \leq s_0$ and $A_i^2 \in (F'_i)_0$ if $s_0 + 1 \leq i \leq s$. For $1 \leq i \leq s_0$, the conjugacy of A_i is determined by $\mu_i(x)$. For $s_0 + 1 \leq i \leq s$, as $F'_i/(F'_i)_0$ is an elementary abelian 2-group with a nondegenerate bimultiplicative function m'_i and $(F'_i)_0$ is a one-dimensional torus, we can modify A_i to make it conjugate to $I_{2^k, 2^k}$ or J_{2^k} . Inductively, we construct elements x_1, \dots, x_{2k} of $H_{F'}$ generating $H_{F'}/\ker m'$ with $m(x_{j_1}, x_{j_2}) = 1$ if and only if $\{j_1, j_2\} = \{2j-1, 2j\}$ for some $1 \leq j \leq k$, such that the conjugacy class of the tuple (x_1, \dots, x_{2k}) is determined by $\mu_1, \mu_2, \dots, \mu_{s_0}$. In this way $H_{F'} = \langle F'_0, B_{F'}, [iI], x_1, \dots, x_{2k} \rangle$ is determined accordingly. Since

$$p^{-1}(F') \subset \overline{G}^{B_F} = (\mathrm{U}(n_1) \times \cdots \times \mathrm{U}(n_s)) \rtimes \langle \tau \rangle.$$

Fixing $H_{F'}$, the conjugacy class of an element $\eta \in F' - H_{F'}$ as an element of $p(\overline{G}^{B_F})$ is determined uniquely modulo $H_{F'}$. Therefore the conjugacy class of F' is determined. \square

Remark 5.1. From the above proof, one sees that F always contains an automorphism of order 2 or 4. There should be a criterion of when F contains an outer involution in terms of $\vec{\mu} = (\mu_1, \dots, \mu_{s_0})$.

The following two propositions are analogues of Propositions 3.1 and 3.3, which could also be proved along the same line.

Proposition 5.2. *If F is a closed abelian subgroup of G satisfying the condition $(*)$ and being not contained in G_0 , then there exists integers $k \geq 0$ and $s_0, s_1 \geq 0$ such that $n = 2^k s_0 + 2^{k+1} s_1$, $\dim F_0 = s_1$, $\mathrm{rank}(H_F/\ker m) = 2k$ and $\mathrm{rank}(\ker m/F_0) \leq \max\{s_0 - 1, 0\}$.*

Proposition 5.3. *If F is a closed abelian subgroup of G satisfying the condition $(*)$, then it is an elementary abelian 2-subgroup if and only if $s_0 = s$ and $\mu_i = \mu_j$ for any $1 \leq i, j \leq s_0$. If F is a maximal abelian subgroup, then $\mathrm{rank}(\ker m/F_0) = \max\{s_0 - 1, 0\}$; if $\mathrm{rank}(\ker m/F_0) = \max\{s_0 - 1, 0\}$ and $(s_0, s_1) \neq (2, 0)$, then it is a maximal abelian subgroup.*

Given a maximal abelian subgroup F' of \tilde{G} , by Proposition 5.1, we associate a function $m' : H_{F'} \times H_{F'} \rightarrow \{\pm 1\}$, integers $k, s, s_0, s_1 = s - s_0$ with $n = 2^k s_0 + 2^{k+1} s_1$ and maps $\mu_1, \mu_2, \dots, \mu_{s_0} : H_{F'}/\ker \nu' \rightarrow \{\pm 1\}$. Denote by $F'_{k, m', \vec{\mu}}$ a maximal abelian subgroup of \tilde{G} like this and $F_{k, m', \vec{\mu}} = \pi'(F'_{k, m', \vec{\mu}})$ be the corresponding maximal abelian subgroup of G . where $\vec{\mu}$ means the unordered tuple $(\mu_1, \mu_2, \dots, \mu_{s_0})$.

For a map $\mu : H_{F'} \rightarrow \{\pm 1\}$, let a_μ be the number of indices i , $1 \leq i \leq s_0$ such that $\mu_i = \mu$. Denote by $S_{\vec{\mu}} = \prod_\mu S_{a_\mu}$. The following proposition describes the Weyl groups of maximal abelian subgroups of G .

Proposition 5.4. *For Let $F' = F'_{k,m',\vec{\mu}}$ and $F = F_{k,m',\vec{\mu}}$, there is an exact sequence*

$$\begin{aligned} 1 &\longrightarrow \text{Hom}(F'/\ker m', B_{F'}) \rtimes S_{\vec{\mu}} \longrightarrow W(F_{k,m',\vec{\mu}}) \\ &\longrightarrow \text{Aut}(H_{F'}/\ker \nu', m', \vec{\mu}) \times (\{\pm 1\}^{s_1} \rtimes S_{s_1}) \longrightarrow 1. \end{aligned}$$

Proof. There is a natural homomorphism $\phi : W(F') \rightarrow W(F)$. We show that ϕ is surjective and $\ker \phi = \langle \text{Ad}([\frac{1+i}{\sqrt{2}}I]) \rangle$. For $g \in \tilde{G}_0$, suppose that $\pi'(g) \in N_G(F)$. Then, $\pi'(gF'g^{-1}) = \pi'(F')$. That means $\text{Ad}(g)F'$ is another lift of F . By Lemma 5.7, one has $\text{Ad}(g)F' = \text{Ad}([\lambda I])F'$ for some $\lambda I \in Z_n$. Thus $[\lambda I]^{-1}g \in N_{\tilde{G}}(F')$. Therefore ϕ is surjective. For $g \in \tilde{G}_0$, suppose that $\text{Ad}(g) \in W(F')$ and $\phi(\text{Ad}(g)) = 1$. Then, $\pi'(g) \in C_G(F) = F$. Hence $g \in p(Z_n)F'$. We may assume that $g = [\lambda I]$ for some $\lambda \in \text{U}(1)$. Choosing an element $u \in F' - H_{F'}$, thus $[\lambda^2 I] = gug^{-1}u^{-1} \in F'$. Since $F' \cap p(Z_n) = p(\langle iI \rangle)$, one has $\text{Ad}(g) \in \langle \text{Ad}([\frac{1+i}{\sqrt{2}}I]) \rangle$. By this we have

$$W(F) = W(F') / \langle \text{Ad}([\frac{1+i}{\sqrt{2}}I]) \rangle.$$

The induced action of an $w \in W(F')$ on $H_{F'}/\ker \nu'$ preserves m' and $\vec{\mu}$. Hence there is a homomorphism $p : W(F') \rightarrow \text{Aut}(H_{F'}/\ker \nu', m', \vec{\mu}) \times W(F'_0)$, which is apparently a surjective map. Clearly one has $W(F'_0) = \{\pm 1\}^{s_1} \rtimes S_{s_1}$. There is another homomorphism $p' : \ker p \rightarrow W(\ker \nu')$. The action of an $w \in \ker p$ on $B_{F'}$ induces a permutation on the first s_0 components isomorphic to $\text{U}(2^k)$ of \overline{G}^{B_F} and hence a permutation of the indices $\{1, 2, \dots, s_0\}$, denoted by σ . Since w acts trivially on $H_{F'}/\ker \nu'$, one has $\mu_{\sigma(i)} = \mu_i$ for each $1 \leq i \leq s_0$. Therefore $\text{Im } p' \subset \prod_\mu S_{a_\mu}$. Considering the preservation of eigenvalues, one shows that $\ker p' \subset \text{Hom}(F'/\ker m', B_{F'}) \times \langle \text{Ad}([\frac{1+i}{\sqrt{2}}I]) \rangle$. Moreover one can show that

$$\ker p / \langle \text{Ad}([\frac{1+i}{\sqrt{2}}I]) \rangle = \text{Hom}(F'/\ker m', B_{F'}) \rtimes S_{\vec{\mu}}.$$

Therefore we get the exact sequence in the conclusion. \square

REFERENCES

- [AYY] J. An; J.-K. Yu; J. Yu, *On the dimension datum of a subgroup and its application to isospectral manifolds*. J. Differential Geom. **94** (2013), no. 1, 59-85.
- [B] A. Borel, *Sous-groupes commutatifs et torsion des groupes de Lie compacts connexes*. (French) Tohoku Math. J. (2) **13** (1961) 216-240.
- [Bo] N. Bourbaki, *Lie groups and Lie algebras*. Chapters 46. Translated from the 1968 French original by Andrew Pressley. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 2002.
- [BtD] T. Bröcker; T. tom Dieck, *Representations of compact Lie groups*. Translated from the German manuscript. Corrected reprint of the 1985 translation. Graduate Texts in Mathematics, 98. Springer-Verlag, New York, 1995.
- [BK] Y. Bahturin; M. Kochetov, *Classification of group gradings on simple Lie algebras of types A, B, C and D*. J. Algebra **324** (2010), no. 11, 2971-2989.

- [Eld10] A. Elduque, *Fine gradings on simple classical Lie algebras*. J. Algebra **324** (2010), no. 12, 3532-3571.
- [EK] A. Elduque; M. Kochetov, *Weyl groups of fine gradings on simple Lie algebras of types A, B, C and D*. arXiv:1109.3540v1.
- [Gr] R.L. Griess, *Elementary abelian p -subgroups of algebraic groups*. Geom. Dedicata **39** (1991), no. 3, 253-305.
- [HY] J.-S. Huang; J. Yu, *Klein four subgroups of Lie algebra automorphisms*. Pacific J. Math. **262** (2013), no. 2, 397-420.
- [PZ89] J. Patera; H. Zassenhaus, *On Lie gradings. I*, Linear Algebra Appl. **112** (1989), 87-159.
- [Yu] J. Yu, *Elementary abelian 2-subgroups of compact Lie groups*. Geom. Dedicata **167** (2013), no. 1, 245-293.

Jun Yu

School of Mathematics,
Institute for Advanced Study,
Einstein Drive, Fuld Hall,
Princeton, NJ 08540, USA
email:junyu@math.ias.edu.